STATISTICAL LAW OF STREAM NUMBERS[1]

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ABSTRACT

The statistical nature and remarkable generality of Horton's law of stream numbers suggest the speculation that the law of stream numbers arises from the statistics of a large number of randomly merging stream channels in somewhat the same fashion that the law of perfect gases arises from the statistics of a large number of randomly colliding gas molecules. The fact that networks with the same number of first-order Strahler streams are comparable in topological complexity suggests equating "randomly merging stream channels" with a topologically random population of channel networks, defined as a population within which all topologically distinct networks with given number of first-order streams are equally likely. In a topologically random population the most probable networks approximately obey Horton's law but exhibit certain systematic deviations. For networks with given number of first-order streams, the most probable network order is that which makes the geometric mean bifurcation ratio closest to 4. For networks with both order and number of first-order streams specified, the most probable networks have the property that the bifurcation ratio of the second-order streams is always close to 4 and, hence, that the bifurcation ratios respectively decrease, remain unchanged, or increase with order and the corresponding curves on the Horton diagram are respectively concave upward, straight, or concave downward according as the geometric mean bifurcation ratio is less than, equal to, or greater than 4. Statistical comparison of these properties with 172 published sets of stream numbers strongly supports the conclusion that, as speculated, populations of natural channel networks developed in the absence of geologic controls are topologically random and, hence, that the law of stream numbers is indeed largely a consequence of random development of channel networks according to the laws of chance.

INTRODUCTION

In a paper now famous among students of geomorphology and hydrology Robert E. Horton in 1945 published two remarkable laws of drainage composition connecting the numbers and the lengths of the streams of different order in a river network. Horton's law of stream numbers states that "the numbers of streams of different orders in a given drainage basin tend closely to approximate an inverse geometric series in which the first term is unity and the ratio is the bifurcation ratio" (Horton, 1945, p. 291; see also 1932, p. 356). This law was chosen for investigation in the present paper because it involves only the relative arrangement of the streams in a channel network without regard to their length, shape, or orientation, and is therefore more amenable to mathematical analysis than the law of stream lengths and similar laws subsequently proposed by others (Schumm, 1956, p. 604, 606, 607; Morisawa, 1962, p. 1033–1034, 1035).

Horton's law of stream numbers is essentially a statistical relationship, inasmuch as the phrase, "tend closely to approximate an inverse geometric series," indicates both the central values and the dispersion of the stream numbers. In other words, Horton's law indicates the distribution of natural river networks among the possible sets of stream numbers, the most probable networks according to Horton being those with stream numbers close to inverse geometric series. It is thus a particular example of the general notion of a *law of stream numbers*, defined in this paper as the statistical distribution of a natural or theoretical population of channel networks among the possible sets of stream numbers determined by a specified system of stream ordering. Despite appearances, however, Horton's law is not fully quantitative, for it gives neither the exact distributions nor even the central values.

Horton (1945, p. 303) believed that his law of stream numbers is relatively insensitive to lack of isotropy and homogeneity in the bedrock, but that departures from the law "will, however, be observed, and if other

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conditions are normal these departures may in general be ascribed to effects of geologic controls." Since 1945 investigators working in many different areas have confirmed this insensitivity, showing that both the geometric-series form and the bifurcation ratio are characterized by considerable independence of the detailed geomorphic processes at work in any particular channel network, which in turn implies very general basic causes.

These characteristics of Horton's law suggest the speculation that in the absence of *geologic controls*, defined in this paper as systematic inhomogeneity or anisotropy in bedrock or environment, the population of natural channel networks is governed primarily by the general tendency of erosional processes to produce arborescent networks and secondarily, or perhaps not at all, by local environmental factors. This leads naturally to the further speculation that the law of stream numbers arises from the statistics of a large number of randomly merging stream channels in somewhat the same fashion that the law of perfect gases arises from the statistics of a large number of randomly colliding gas molecules.

The object in this paper is to give substance to these heuristic ideas and to explore some of their theoretical ramifications by elucidation of the mathematical properties of Horton's law and by examination of a particular definition of "randomly merging stream channels."

HORTON'S LAW OF STREAM NUMBERS

In mathematical terms Horton's law of stream numbers states that

$$H_\omega \approx h_\omega , \qquad (1a)$$

where $\approx$ denotes approximate equality, H_ω is the number of Horton streams of order ω in a given drainage basin, and h_ω is the corresponding term in the geometric series defined by

$$h_\omega = R_H^{\Omega-\omega} , \qquad (1b)$$

in which R_H is the Horton bifurcation ratio and Ω is the order of the basin, which is by definition equal to the order of the highest-order stream in the basin.

In the system of classifying streams according to order devised by Horton (1945, p. 281),

> unbranched fingertip tributaries are always designated as of order 1, tributaries or streams of the 2d order receive branches or tributaries of the 1st order, but these only; a 3d order stream must receive one or more tributaries of the 2d order but may also receive 1st order tributaries. A 4th order stream receives branches of the 3d and usually also of lower orders, and so on.
>
> To determine which is the parent and which the tributary stream upstream from the last bifurcation, the following rules may be used:
>
> (1) Starting below the junction, extend the parent stream upstream from the bifurcation in the same direction. The stream joining the parent stream at the greatest angle is of the lower order. Exceptions may occur where geologic controls have affected the stream courses.
>
> (2) If both streams are at about the same angle to the parent stream at the junction, the shorter is usually taken as of the lower order.

Some workers (Leopold and Miller, 1956, p. 16; Brush, 1961, p. 155; Leopold and Langbein, 1962, p. A15) have followed Horton's system of determining stream orders, but others (Schumm, 1956, p. 602; Melton, 1957, p. 2; Coates, 1958, p. 4; Smith, 1958, p. 999, 1003; Maxwell, 1960, p. 9; Morisawa, 1962, p. 1028), in order to avoid the necessity of subjective decisions inherent in Horton's system, have adopted a somewhat different system introduced by Strahler (1952, p. 1120 n.; see Melton, 1959, for alternative definition). In Strahler's system "the smallest, or 'finger-tip,' channels constitute the first-order segments," somewhat as in Horton's system, but a "second-order segment is formed by the junction of any two first-order streams; a third-order segment is formed by the joining of any two second-order streams, etc."

The application of these two systems to a particular stream network is illustrated in figures 1 and 2.

Stream numbers for a given network determined according to the Strahler system cannot in general be the same as those deter-

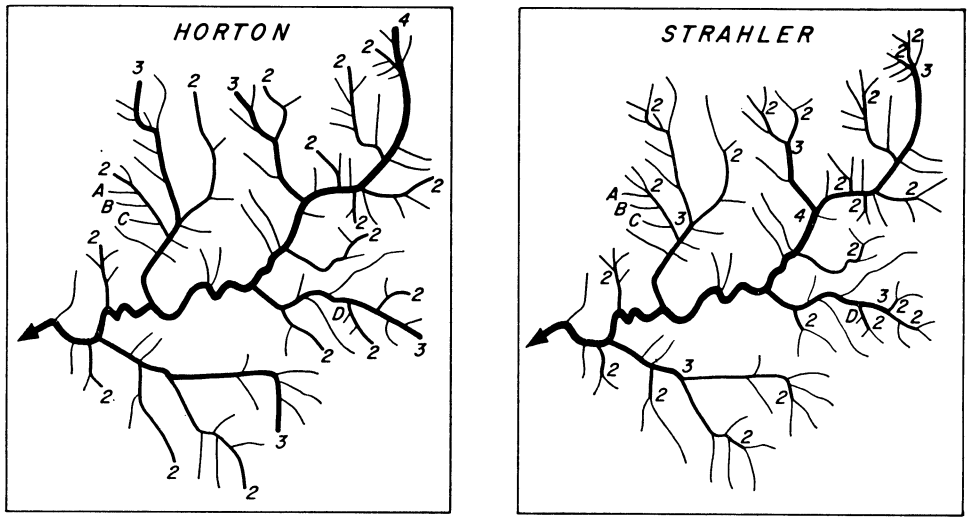

FIG. 1.—Horton and Strahler definitions of stream order applied to channel network of Hightower Creek, upper Hiwassee River, Towns County, northern Georgia (redrawn from Horton, 1945, p. 297). Order indicated by number near upstream end of respective streams. Unnumbered streams are first order. If streams *A*, *B*, *C*, or *D* were actually second order, rather than first, then network would be fifth order, rather than fourth.

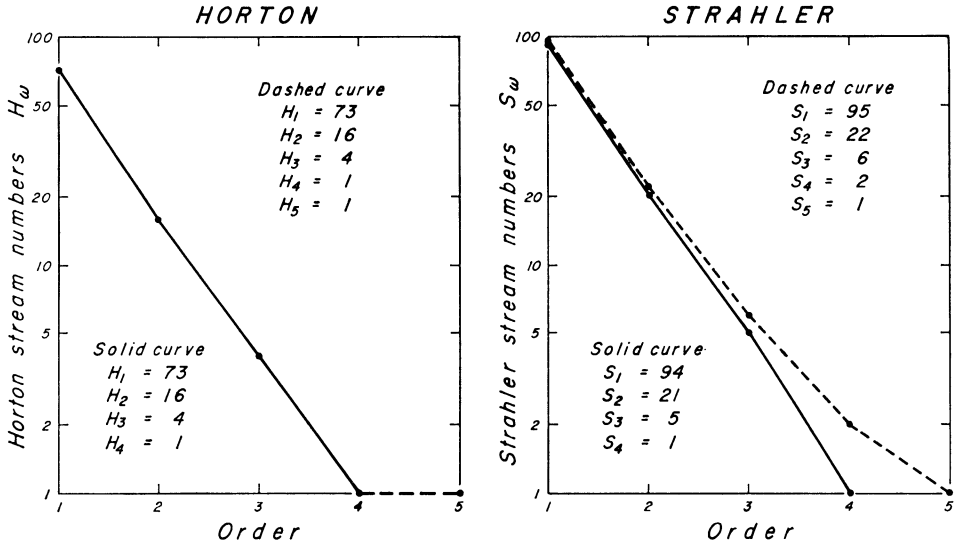

FIG. 2.—Horton diagrams for channel network of Hightower Creek as shown in fig. 1. Solid curves for network as shown. Dashed curves for fifth-order network resulting if one of streams *A*, *B*, *C*, or *D* were actually second order.

mined according to the Horton system; nevertheless, the Strahler stream numbers appear to approximate inverse geometric series (Schumm, 1956, p. 603; Smith, 1958, p. 1002; Maxwell, 1960, p. 61; Morisawa, 1962, p. 1929) differing from the Horton series only in the magnitude of the bifurcation ratios. In analogy with equation (1),

$$S_\omega \approx s_\omega , \qquad (2a)$$

where S_ω is the number of Strahler streams of order ω and s_ω is the corresponding term in the geometric series defined by

$$s_\omega = R_S^{\Omega-\omega} , \qquad (2b)$$

in which R_S is the Strahler bifurcation ratio. Both this relationship and Horton's original one are commonly termed *Horton's law of stream numbers.*

BASIC CONCEPTS AND METHODS

The channels and lakes, if any, upstream of an arbitrarily chosen channel cross section, termed the *outlet* of the network, collectively comprise a *channel network*, also called a *drainage network*, *stream network*, or *river network*. The points farthest upstream in a channel network are termed *sources*. The point of confluence of two channels is a *fork*. The term *link* will refer to a section of channel reaching without intervening forks from either a fork or a source at its upstream end to either a fork or the outlet at its downstream end.

Correct, or at least consistent, identification of sources and channels on maps and air photographs and in the field is of fundamental importance in the study of channel networks. The problems involved have therefore been recognized and investigated by many workers (Horton, 1945, p. 284; Leopold and Miller, 1956, p. 2–4, 16–19; Melton, 1957, p. 1, 7; Morisawa, 1957; Maxwell, 1960, p. 24–25), and will not be discussed further in this paper.

Confluences of more than two channels precisely at a single place are exceedingly rare in nature, and for the purpose of analysis are usually judiciously resolved into two or more forks. Lakes fed by multiple inlets and islands bounded by branching channels pose similar difficulties, however, that are not so easily resolved. Although Horton (1945, p. 289) evidently was not bothered by such difficulties, most workers seem to have avoided them by choosing to study networks without lakes or islands, such as those normally associated with mature topography.

The *Horton bifurcation ratio* R_H is conventionally determined from the slope $-\log R_H$ of the straight line

$$\log h_\omega = (\Omega - \omega) \log R_H \qquad (3a)$$

that passes through $(\Omega, 0)$ and "best fits" the points on a *Horton diagram*, which is a plot of $\log H_\omega$ versus ω. The *Strahler bifurcation ratio* R_S, which can never be less than 2, is found in analogous fashion from a plot of $\log S_\omega$ versus ω, also called a Horton diagram. Maxwell (1960, p. 12), on the other hand, has argued that the "best" straight line should not be required to pass through $(\Omega, 0)$. As either method would lead to the same conclusions, the conventional definition will be used in this paper.

The method of least squares leads straightforwardly to the formula

$$\log R_H = \frac{6}{\Omega(\Omega-1)(2\Omega-1)} \times \sum_{\omega=1}^{\Omega-1} (\Omega-\omega)\log H_\omega \qquad (3b)$$

for R_H. This equation gives the value of R_H for which the root-mean-square deviation

$$D_H = \left[\frac{1}{\Omega}\sum_{\omega=1}^{\Omega-1} (\log H_\omega - \log h_\omega)^2\right]^{1/2} \qquad (3c)$$

is a minimum. Equation (3c) may be written in terms of R_H in the useful form

$$D_H = \left[\frac{1}{\Omega}\sum_{\omega=1}^{\Omega-1} (\log H_\omega)^2 - \tfrac{1}{6}(\Omega-1)(2\Omega-1)(\log R_H)^2\right]^{1/2} . \qquad (3d)$$

The corresponding formulas relating to the Strahler bifurcation ratio may be obtained

by writing R_S, S_ω, s_ω, and D_S, respectively, for R_H, H_ω, h_ω, and D_H in (3).

The unqualified term *bifurcation ratio* or the term *bifurcation ratio of streams of order* ω will refer to B_ω as defined by the equation (see Melton, 1958, p. 44)

$$B_\omega = S_{\omega-1}/S_\omega . \qquad (4a)$$

This ratio, which can never be less than 2, is the average number of Strahler streams of order $\omega - 1$ tributary to each Strahler stream of order ω, and is therefore the Strahler bifurcation ratio obtained by considering only streams of order ω and $\omega - 1$. In terms of B_ω the Strahler stream numbers are given by

$$S_\omega = B_{\omega+1} B_{\omega+2} \ldots B_\Omega , \qquad (4b)$$
$$\omega = 1, 2, \ldots, \Omega - 1 , \quad S_\Omega = 1 ,$$

where Ω is the order of the channel network. An analogous definition may be made in terms of Horton stream numbers (Horton, 1945, p. 280), but will not be used in this paper.

The *geometric mean bifurcation ratio* B, which also can never be less than 2, will be defined by the equation

$$B = (B_2 B_3 \ldots B_\Omega)^{1/(\Omega-1)} = S_1^{1/(\Omega-1)} , \qquad (4c)$$

where Ω is the order of the channel network. It is equal to the Strahler bifurcation ratio which would be computed from the slope of the straight line that passes through the two end points $(1, \log S_1)$ and $(\Omega, 0)$ on a Horton diagram. The quantity B, therefore, though closely similar to the Strahler bifurcation ratio R_S, does not depend upon either Horton's law or "best fits." In a channel network obeying Horton's geometric-series law, B would be exactly equal to R_S.

Horton (1945, p. 290) believed that R_H is greater "for hilly, well-dissected drainage basins than for rolling basins," being about 3 to 4 for the former and only about 2 for the latter. Strahler has found, however, that R_S, which should be closely correlated with R_H and therefore with relief, is instead completely uncorrelated with relief (1952, p. 1136) and, further, that it "is highly stable and shows a small range of variation from region to region or environment to environment, except where powerful geologic controls dominate," its range being from 3 to 5 but its usual value being about 4 (1957, p. 914). Histograms of R_H and R_S are given for comparison in figure 3. In a different type of investigation Morisawa (1962, p. 1042, table 6) has shown, moreover, that R_S is not highly correlated with peak flow, with mean runoff, or with mean annual discharge from watersheds in the Appalachian Plateau, the correlation coefficients ranging from .10 to .26. Thus, although Horton's law of stream numbers is intimately and intricately connected with the law of stream lengths and with other geometric properties of drainage systems (Melton, 1958, p. 44–46, 53), the bifurcation ratio is largely independent of the other geomorphic and hydrologic variables and of the environment.

STREAM ORDERS AND STREAM NUMBERS

According to my interpretation the *Horton order* of a link in a stream network may be determined by means of the following four rules: (1) a set of contiguous links of order ω extending from a given fork upstream to a source is termed a *provisional Horton stream* of order ω; (2) each link originating at a source is initially considered to be a provisional Horton stream of order 1; (3), if two provisional Horton streams of the same order ω join, then the resultant link downstream has order $\omega + 1$ and that tributary stream is reclassified as order $\omega + 1$, which either (a) enters the junction in a direction more nearly parallel to the head of the resultant link, or (b), in case both enter at about the same angle, is longer, or (c) is for some other reason, such as the presence of geologic controls, considered to be the parent stream; and (4), if two provisional Horton streams of different order join, then the resultant link downstream has order equal to that of the tributary of higher order.

Recasting Strahler's definition in parallel terms, the *Strahler order* of a link in a stream network may be determined by means of the

following four rules: (1) a set of contiguous links of order ω extending from a given fork upstream to a fork where two links of order $\omega - 1$ join is termed a *provisional Strahler stream* of order ω; (2) each link originating at a source is initially considered to be a provisional Strahler stream of order 1; (3), if two provisional Strahler streams of the same order ω join, then the resultant link downstream has order $\omega + 1$; and (4), if two provisional Strahler streams of different order join, then the resultant link downstream has order equal to that of the tributary of higher order. The Strahler definition of stream order thus differs from the Horton definition only in the omission of the somewhat arbitrary reclassification in rule (3) and in the revision this necessitates in rule (1).

In both systems, after the order of all links has been determined, the qualification *provisional* is dropped, and all streams except the one of highest order are considered *complete*, because they debouch into streams of higher order. The highest order stream may be either complete or incomplete, though it is often arbitrarily considered to be complete. A network whose highest order stream is complete is by extension a *complete network* (Strahler, 1952, p. 1120; Melton, 1957, p. 2; Morisawa, 1962, p. 1028–1029), and the converse.

The order of a channel network or drainage basin is defined as being the order of its highest order stream. The Strahler order of any network is fixed by the arrangement of links within the network itself. The Horton order of an incomplete network, on the other hand, depends upon the arrangement of links in the larger network of which the given network is part. If a given network is complete or arbitrarily considered to be complete, its Horton and Strahler orders will be equal.

In either system the effect of disregarding all streams of the lowest α orders is to diminish the apparent order of the remaining

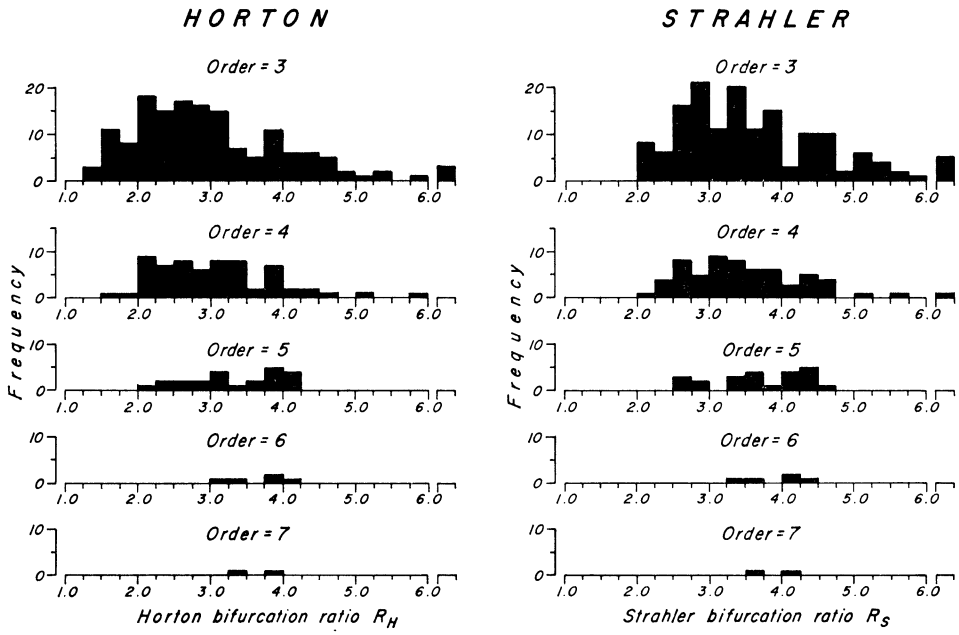

FIG. 3.—Histograms of Horton and Strahler bifurcation ratios R_H and R_S, respectively, for 246 natura networks for which usable sets of stream numbers have been published (references in table 1). Lower limit included and upper limit excluded in each interval. Ratios equal to or greater than 6.0 lumped in separate interval at right.

streams by α; thus, in such cases streams of actual order ω appear to be of order $\omega - \alpha$. In certain cases, as illustrated in figure 1, omission or addition of a single first-order stream can change the apparent order of the entire network (Maxwell, 1960, p. 24–25).

The Strahler system of stream ordering, unlike the Horton system, is purely topological, for it refers only to the interconnections and not to the lengths, shapes, or orientations of the links comprising a network. Surprisingly, however, the "subjective decisions inherent in Horton's system" have no influence upon the resultant stream numbers. In fact, even when the criteria used are completely arbitrary, the Horton stream numbers associated with a network that is complete or arbitrarily considered to be complete are entirely unaffected by the reclassification in rule (3). This conclusion follows from the observation that every Horton stream of order ω consists of ω Strahler streams, one of each order from 1 to ω, laid end to end in sequence, and every Strahler stream is part of one and only one Horton stream, as may be seen in figure 1; hence,

$$S_\omega = \sum_{\alpha=\omega}^{\Omega} H_\alpha, \quad \omega = 1, 2, \ldots, \Omega, \quad (5a)$$

and, conversely,

$$H_\omega = S_\omega - S_{\omega+1}, \qquad \Omega = 1, 2, \ldots, \Omega - 1, \qquad H_\Omega = 1, \quad (5b)$$

from which the assertion results directly, inasmuch as the S_ω are unambiguously defined. Equations (5) enable conversion of stream numbers from one system to the other without the lengthy classification and counting otherwise required; and they show that in any channel network which is complete or arbitrarily considered to be complete the number of first-order Strahler streams is equal to the total number of Horton streams.

COMPARISON OF THE TWO LAWS

A number of authors (Schumm, 1956, p. 603; Melton, 1957, p. 5; Smith, 1958, p. 1002; Maxwell, 1960, p. 12; Morisawa, 1962, p. 1029) have either stated or implied that the law expressed by (2) is Horton's law of stream numbers. If the two laws (1) and (2) were in fact equivalent, the series terms in (1b) and (2b) would satisfy the relations (5). Direct substitution in (5b) leads to the set of equations

$$R_H^{\Omega-\omega} = R_S^{\Omega-\omega} - R_S^{\Omega-\omega-1}, \qquad \Omega = 1, 2, \ldots, \Omega - 1, \quad (6)$$

whose only real solution is $R_H = 0$, $R_S = 1$; hence, inasmuch as R_S can never be less than 2, (1) and (2) cannot be mathematically equivalent, but are different physical laws. The system of determining stream orders must therefore be explicitly and unambiguously specified in every investigation involving Horton's law of stream numbers, a point which has not always been appreciated, as Bowden and Wallis have pointed out (1964, p. 768).

The root-mean-square deviation D_S is less than D_H in 210 of the 246 natural networks for which usable sets of stream numbers have been published (references in table 1). Although this sample is undoubtedly not strictly random, at least it consists entirely of networks selected by other authors for other purposes, and it comprises all the data given in each reference except for those few unusable cases in which, in descending order of frequency, stream numbers were given only as points on insufficiently detailed Horton diagrams, the stated total number of Horton streams disagreed with the actual sum, stream numbers from several networks were summed or averaged, or one of the ratios B_ω was less than 2.

Detailed data and references are presented in figure 4 and table 1 (logarithms to base 10 used).

If p is the probability that in a randomly chosen natural network the root-mean-square deviation D_S will be less than the root-mean-square deviation D_H, then the probability $P(n; N, p)$ that $D_S < D_H$ in exactly n of N randomly chosen networks is

$$P(n; N, p) = \binom{N}{n} p^n (1 - p)^{N-n}. \quad (7)$$

Hence, on the undoubtedly shaky assumption that the present sample of 246 networks is random, the maximum-likelihood estimate of p (Mood, 1950, p. 152–161) is $p_m = 210/246 = 0.855$, and the 0.95 confidence interval for p (Mood, 1950, p. 233–237) is $0.811 < p < 0.899$. More detailed results are presented in table 1. In the cases where N was 2, 5, 23, and 64, respectively, the exact equations (Mood, 1950, p. 234) and *Tables of the Cumulative Binomial Probability Distribution* (Harvard University, Staff of the Computation Laboratory, 1955) were used. In the cases where N was 152 and 246, respectively, the normal approximation (Mood, 1950, p. 236) was used. Except for networks of order 6 and 7, for which insufficient data are available, these results indicate that the Strahler stream numbers for the population sampled generally approximate inverse geometric series more closely than do the corresponding Horton stream numbers.

SYSTEMATIC DEVIATIONS

Both Schumm (1956, p. 603) and Maxwell (1960, p. 12) have called attention to the fact that many of their curves of $\log S_\omega$ versus ω are distinctly concave upward. An example based on stream numbers given by Morisawa (1962, p. 1036) is shown in figure 5. This concavity suggests replacing (2) by

$$S_\omega \approx c_\omega , \tag{8a}$$

$$\log c_\omega = [(\Omega - \omega)/(\Omega - 1)] \times [1 - a(\omega - 1)] \log S_1 , \tag{8b}$$

where the parameter a, like the bifurcation ratio, is adjusted to give the "best fit" to a plot of Strahler stream numbers on a Horton diagram. Equation (8b) represents a parabolic curve that passes through (1, $\log S_1$) and (Ω, 0) on the Horton diagram and is concave upward, straight, or concave downward when a is positive, zero, or negative, respectively. Thus, inasmuch as a will generally be greater for Horton stream numbers

TABLE 1

ANALYSIS OF PUBLISHED STREAM NUMBERS*

	Order					
	All	3	4	5	6	7
Networks	246	152	64	23	5	2
$D_S < D_H$	210	125	58	20	5	2
$D_C < D_S$	241	152	62	20	5	2
$D_C < D_H$	244	151	63	23	5	2
Mean D_H	0.140	0.129	0.162	0.150	0.115	0.184
Mean D_S	0.068	0.061	0.082	0.080	0.056	0.106
Mean D_C	0.013	0.000	0.026	0.044	0.040	0.099
Estimate p_m	0.855	0.823	0.907	0.870	1.000	1.000
Confidence interval for p	0.811–0.899	0.763–0.883	0.827–0.966	0.720–0.973	0.478–1.000	0.158–1.000
Confidence	0.95	0.95	0.95	0.95	0.95	0.95
Mean a	0.128	0.160	0.095	0.042	0.013	0.010
Median a	0.110	0.143	0.110	0.054	0.015	0.010
Confidence interval for a	0.084–0.139	0.139–0.224	0.075–0.127	0.001–0.073	0.012–0.039	0.010–0.011
Confidence	0.95	0.95	0.97	0.96	0.97	0.50

* Data from Horton (1945, p. 287–288, 290, 297, 302, 306), Leopold and Miller (1956, p. 17), Schumm (1956, p. 606), Melton (1957, table 2, facing p. 88), Coates (1958, p. 51, 53), Smith (1958, p. 1003), Maxwell (1960, p. 71–83), and Morisawa (1962, p. 1036–1037).

than for Strahler stream numbers, it is a suitable measure for investigation of the concavity noted by Schumm and Maxwell.

The method of least squares leads to the formula

$$a = \frac{5}{(\Omega^2 - 2\Omega + 2)} \times \left[\frac{\Omega - 1}{2} - \frac{6}{\Omega(\Omega - 2)\log S_1} \times \sum_{\omega=2}^{\Omega-1} (\Omega - \omega)(\omega - 1)\log S_\omega\right] \quad (9a)$$

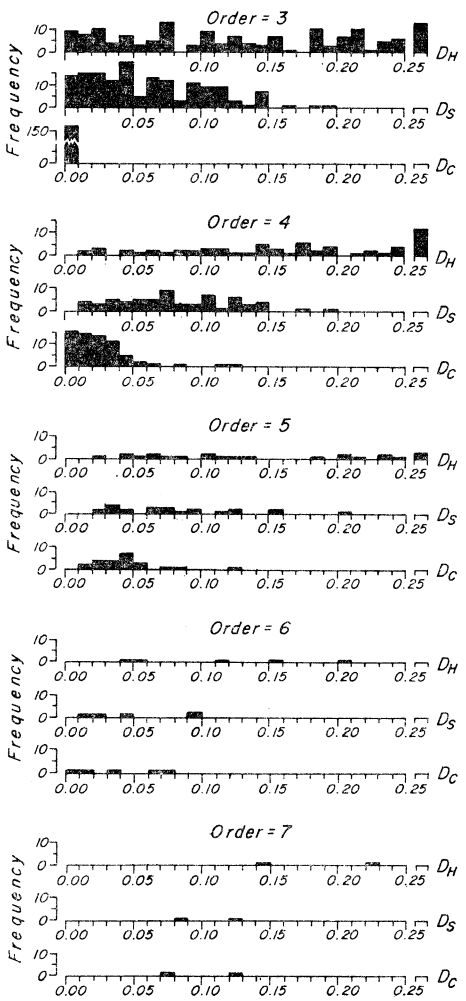

FIG. 4.—Histograms of root-mean-square deviations D_H, D_S, and D_C. Lower limit included and upper limit excluded in each interval. Deviations equal to or greater than 0.25 lumped in separate interval at right.

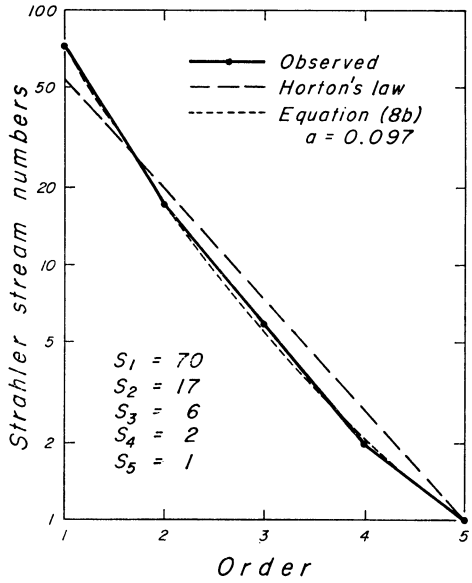

FIG. 5.—Horton diagram for Home Creek, 3 km. (2 mi.) east of New Philadelphia, Tuscarawas County, eastern Ohio (data from Morisawa, 1962, p. 1036), showing Horton's law (long dashes) and law (8b) (short dashes).

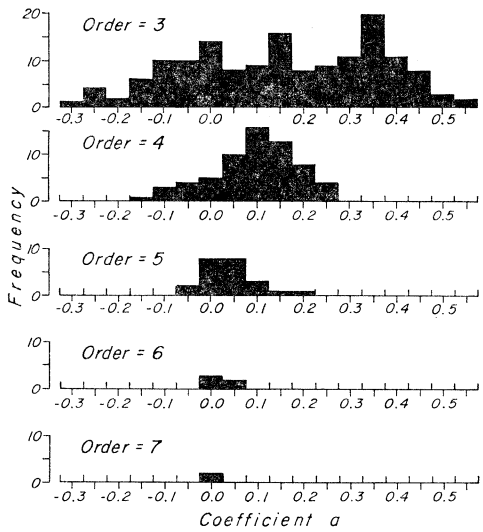

FIG. 6.—Histograms of parameter a in law (8b). Lower limit included and upper limit excluded in each interval. Positive a corresponds to Horton diagrams that are concave upward, and the converse.

for the value of a giving the "best fit" in the sense that the root-mean-square deviation D_c is a minimum. The formula for D_c analogous to (3d) is

$$D_c = \left[\frac{1}{\Omega}\sum_{\omega=2}^{\Omega-1}(\log S_\omega)^2 - \frac{2\log S_1}{\Omega(\Omega-1)}\sum_{\omega=2}^{\Omega-1}(\Omega-\omega)\log S_\omega + \frac{(\Omega-2)(\log S_1)^2}{30\Omega(\Omega-1)}\{5(2\Omega-3) - a^2\Omega(\Omega^2-2\Omega+2)\}\right]^{1/2}, \quad (9b)$$

which is easy to evaluate because the two sums are byproducts of the computation of R_S and D_S. Results of numerical computations for the 246 networks previously analyzed are presented in table 1 and figures 4 and 6. The multimodal character of the histogram for third-order networks in figure 6 is due to the fact that for small networks the permissible values of a are more widely spaced than the class intervals.

Except for the data of Horton (1945) the sample mean values of a summarized in table 1 are consistently positive, showing that the curves for the 246 networks analyzed do indeed tend to be slightly concave upward. Unfortunately, an estimate and confidence interval for the population mean cannot be given because the distribution of a is unknown. An estimate and confidence interval for the population median, however, can be obtained by distribution-free methods (Mood, 1950, p. 387–390). The population median α as estimated by the sample median for the 246 networks is 0.110, and the 0.95 confidence interval for α is $0.084 < \alpha < 0.139$, again assuming a random sample. As before, the exact equations (Mood, 1950, p. 388–389) were used when N was 2, 5, 23, and 64, whereas the normal approximation (Mood, 1950, p. 389) was used when N was 152 and 246. Both the mean and median values of a decrease monotonically with increasing network order, thereby tending to substantiate the speculation to this effect originally made by Smith (1958, p. 1002) on the basis of only five networks. Evidently the law of stream numbers for natural channel networks is not quite the simple relationship proposed by Horton.

"RANDOMLY MERGING STREAM CHANNELS"

Perhaps the most natural and certainly the most direct realization of "randomly merging stream channels" for theoretical purposes is achieved by graphical construction according to rules whose action is governed by chance events such as the drawing of shuffled cards or random numbers, the particular rules and probabilities used implicitly defining the theoretical population generated. A fine example of a network produced by this method has been published by Leopold and Langbein (1962, p. A18; reproduced by Leopold, Wolman, and Miller, 1964, p. 418), although unfortunately the authors omitted the rules for avoiding or eliminating closed loops and dead-end spirals. A slight modification of their method has already been partially automated on a small-scale digital computer by Schenck (1963).

The graphical method has a number of advantages in addition to being natural and direct: it is relatively nonabstract and nonmathematical; it involves the same methods of sampling and analysis as natural networks; and, perhaps most important, by simple manipulation of pattern and probabilities, it is capable of simulating the effects of geologic controls, the evolution of channel networks, and possibly even the results of environmental variations. On the other hand, it has several major disadvantages: it requires rather highly simplified basic stream patterns in order to be practical; it inherently involves Monte Carlo methods which not only make difficult the control of parameters, such as the number of first-order streams or the order of the network, but also, as will be shown, entail almost astronomical numbers of trials because of the vast number of possible cases even for relatively small networks; and, finally, it does not directly provide the quantities of interest, such as stream order and number, which must be obtained by the same la-

borious methods that are applied to natural networks.

Because of these latter characteristics the graphical method is impractical for computing the probability of occurrence of a network with a given set of stream numbers or other arbitrarily specified parameters; hence, it is unsuited for detailed investigation of the law of stream numbers for the theoretical population of networks implicitly defined by its rules and probabilities. Moreover, it unavoidably encompasses network properties that are not explicitly involved in the Strahler system of stream ordering, which is purely topological, or in the law of stream numbers, which is entirely enumerative. This paper therefore adopts an alternative definition of "randomly merging stream channels," which involves only network topology and simple enumeration.

For simplicity and convenience in the remainder of this paper the Strahler system of stream ordering will be used exclusively.

TOPOLOGICALLY RANDOM POPULATION

In any network such as the usual channel network associated with mature topography in which one and only one path exists between any two points and in which every link either joins at each end to two other links or connects at one end to two other links and terminates at the opposite end in a source or, in one case, the outlet, the number of links l and the number of forks f are related to the number of sources n by the equations

$$l = 2n - 1 , \qquad (10a)$$

and

$$f = n - 1 , \qquad (10b)$$

which follow by induction from the facts that networks with only one source have only one link and no forks and that addition of a source to any network increases the number of links by two and the number of forks by one. These relationships were first explicitly stated by Melton (1959, p. 345), although they result directly from two general topological theorems respectively deduced in the famous paper on the Königsberg bridges by Euler (1741) and in a little-known work on topology by Listing (1847; also 1862; see Cayley, 1873, for English summary, reprinted in 1895, p. 540–547). The conditions stated at the beginning of this paragraph, which in practice are not particularly restrictive, define the idealized concept of channel network assumed in the remainder of this paper.

Equations (10) and (5) show that channel networks with equal numbers of links have equal numbers of forks, sources, Horton streams, and first-order Strahler streams; and are therefore comparable in topological complexity, as illustrated in figure 7. This suggests equating "randomly merging stream channels" with a *topologically random* population of channel networks, defined as a population within which all topologically distinct networks with given number of links are equally likely. *Topologically distinct* networks, illustrated in figure 7, in contradistinction to *topologically identical* networks, illustrated in figure 8, are those whose schematic map projections cannot be continuously deformed and rotated in the plane of projection so as to become congruent.

The fundamental geomorphological hypothesis that emerges from these strictly heuristic and somewhat abstract considerations is that in the absence of geologic controls a natural population of channel networks will be topologically random. Though intuitively attractive this hypothesis is by no means self-evident, for the possibility exists that certain of the possible topologically distinct networks might be statistically promoted or inhibited, not by gross geologic controls, but by the subtle interaction of the links in networks, or as Playfair put it (1802, p. 102) the "nice adjustment," mediated by the interdependence of slope and channel processes. Justification of the hypothesis therefore ultimately rests upon comparison with natural networks, which for lack of other data means statistical analysis of published sets of stream numbers, hence the emphasis in this paper.

Far more than this, however, is latent in the definition of topologically random chan-

nel networks. It can predict, without any adjustable parameters, not only the law of stream numbers as shown in this paper but also such quantities as the probability that a link chosen at random will drain a basin having a given order or a given number of links. Thus, it can supply both the basic distributions and the sampling requirements for meaningful statistical analysis of any of the enumerative properties of natural networks; it provides a mathematically simple and well-defined model for the deeper elucidation of the topological properties of natural networks; and, perhaps most important, it constitutes the necessary starting point for derivation of the non-enumerative properties of drainage basins, such as the law of stream lengths.

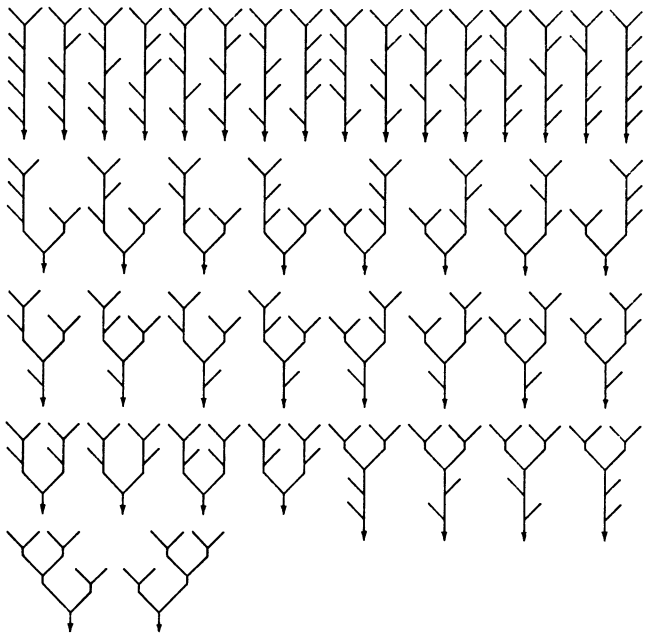

FIG. 7.—Schematic diagrams of the $N(6) = 42$ topologically distinct channel networks with $l = 11$ links and $n = 6$ first-order Strahler streams or six Horton streams. In a topologically random population these networks would all be equally likely. Top row shows the possible second-order networks, for which $\Omega = 2$, $B = 6.00$, $N(6;2) = 16$, $p(6;2) = 16/42 = 0.381$, $N[6,1] = 16$, and $p[6,1] = 16/16 = 1.000$. Bottom four rows show the possible third-order networks, for which $\Omega = 3$, $B = 2.45$, $N(6;3) = 26$, $p(6;3) = 26/42 = 0.619$, $N[6,2,1] = 24$, $p[6,2,1] = 24/26 = 0.923$ (first three of the four rows), $N[6,3,1] = 2$, and $p[6,3,1] = 2/26 = 0.077$ (last row). Arrowhead indicates outlet in each diagram.

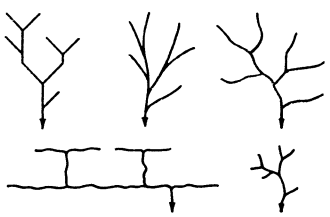

FIG. 8.—Topologically identical channel networks. Arrowhead indicates outlet. Note that network topology, stream pattern, and drainage density, though perhaps physically connected, are mathematically distinct concepts.

COMBINATORIAL PROPERTIES

The number of topologically distinct channel networks having n sources, denoted by $N(n)$, may be computed from the recursion formula

$$N(n) = \sum_{i=1}^{n-1} N(i)\,N(n-i), \qquad N(1) = 1, \quad n = 2, 3, \ldots . \tag{11}$$

The general term in the sum is the number of topologically distinct networks with n

sources formed by combination of the $N(i)$ distinct networks having i sources with the $N(n-i)$ distinct networks having $n-i$ sources in all possible pairs. The result follows by summation over all permissible values of i.

Similarly, the number of topologically distinct channel networks of order Ω having n sources, denoted by $N(n;\Omega)$, may be computed from the recursion formula

$$N(n;\Omega)=\sum_{i=1}^{n-1}\Big[N(i;\Omega-1)\times N(n-i;\Omega-1)+2N(i;\Omega)\times\sum_{\omega=1}^{\Omega-1}N(n-i;\omega)\Big], \quad (12)$$

$$N(1;1)=1\,,\quad N(n;1)=0\,,$$
$$N(1;\Omega)=0\,,\qquad n=2,\,3,\,\ldots\,,$$
$$\Omega=2,\,3,\,\ldots\,.$$

The first product in the general term of the sum on i accounts for the creation of networks of order Ω by combination of two of order $\Omega-1$; and the second accounts for the combination of networks of order Ω with those of lower order.

The problem of finding a simple closed expression for $N(n)$ was first solved by Cayley (1859; reprinted in 1891, p. 112–115), whose solution, after slight manipulation and conversion of notation, is

$$N(n)=\frac{1}{l}\Big[\frac{l(l-1)\ldots(l-n+1)}{n(n-1)\ldots 1}\Big]=\frac{1}{l}\binom{l}{n}, \quad (13)$$

in which $l=2n-1$ as before.

The number of topologically distinct channel networks of order Ω having n_1, $n_2,\ldots,n_{\Omega-1}$, 1 streams of order 1, 2, ..., $\Omega-1$, Ω, respectively, denoted by $N[n_1, n_2,\ldots,n_{\Omega-1},1]$, is given by the formula

$$N[n_1,n_2,\ldots,n_{\Omega-1},1]=\prod_{\omega=1}^{\Omega-1}2^{(n_\omega-2n_{\omega+1})}\binom{n_\omega-2}{n_\omega-2n_{\omega+1}}. \quad (14)$$

The general term of the product is the number of topologically distinct ways that the n_ω streams of order ω may be arranged as tributaries to the streams of higher order in the network. Of these n_ω streams $2n_{\omega+1}$ must join in pairs to form the $n_{\omega+1}$ streams of order $\omega+1$, leaving $r_\omega=n_\omega-2n_{\omega+1}$ remaining streams of order ω tributary to the $l_{\omega+1}=2n_{\omega+1}-1$ links of order $\omega+1$ or higher present before the lower-order streams are added to the network. The binomial coefficient in (14) is the number of distinct ways that the r_ω streams may be divided among the $l_{\omega+1}$ links (Riordan, 1958, p. 6–7). The factor $2^{r\omega}$ is the number of distinct patterns in which the remaining streams may join these links, inasmuch as each must enter from either the right or the left. The general term in (14) is independent of the arrangement of the higher-order links; hence, the number $N[n_1, n_2, \ldots, n_{\Omega-1}, 1]$ is simply the product of the individual terms for all orders from $\Omega-1$ down to 1.

The set of stream numbers $[n_1, n_2, \ldots, n_{\Omega-1}, 1]$ is subject to the restriction, implied by the definition of stream order, that every network must contain at least two streams of order $\omega-1$ for each stream of order ω, for $\omega=2,3,\ldots$. This is equivalent to the restriction

$$r_\omega=0,\,1,\,\ldots\,. \quad (15a)$$

When $n=n_1$ is given, the r_ω must satisfy the equation

$$r_1+2r_2+\ldots+2^{\Omega-2}r_{\Omega-1}=n-2^{\Omega-1}\,. \quad (15b)$$

A unique set $[r_1, r_2, \ldots, r_{\Omega-1}]$ exists for each set $[n_1, n_2, \ldots, n_{\Omega-1}, 1]$, and the converse; therefore, the number of different sets of stream numbers with n and Ω given is the same as the number of solutions in integers of (15b) subject to the restriction (15a), which in turn is the number $D(n;\Omega)$ of partitions of $n-2^{\Omega-1}$ into parts $1, 2, \ldots, 2^{\Omega-2}$, each of which may appear zero, one, or more times. Riordan (1958, p. 117–123) has discussed various methods of calculating this number, the simplest of which leads to the useful recursion formula

$$
\begin{aligned}
D(n;\Omega) &= D(n-2^{\Omega-2};\Omega-1) \\
&\quad + D(n-2^{\Omega-2};\Omega), \\
&\quad n = 2^{\Omega-1},\ 2^{\Omega-1}+1,\ \ldots, \\
D(n;\Omega) &= 0, \qquad (16) \\
&\quad n = 1, 2, \ldots, 2^{\Omega-1}-1, \\
&\quad \Omega = 2, 3, \ldots, \\
D(1;1) &= 1,\ D(n;1) = 0, \\
&\quad n = 2, 3, \ldots .
\end{aligned}
$$

Horton's law, as shown in figure 9. In other words, inherent in the definition of stream order is the corollary that no channel network, indeed no arborescent network of any kind, can depart indefinitely far from Horton's geometric-series law.

LAW OF STREAM NUMBERS

The law of stream numbers for topologically random channel networks is readily found from the combinatorial properties just derived. The procedure is to divide the num-

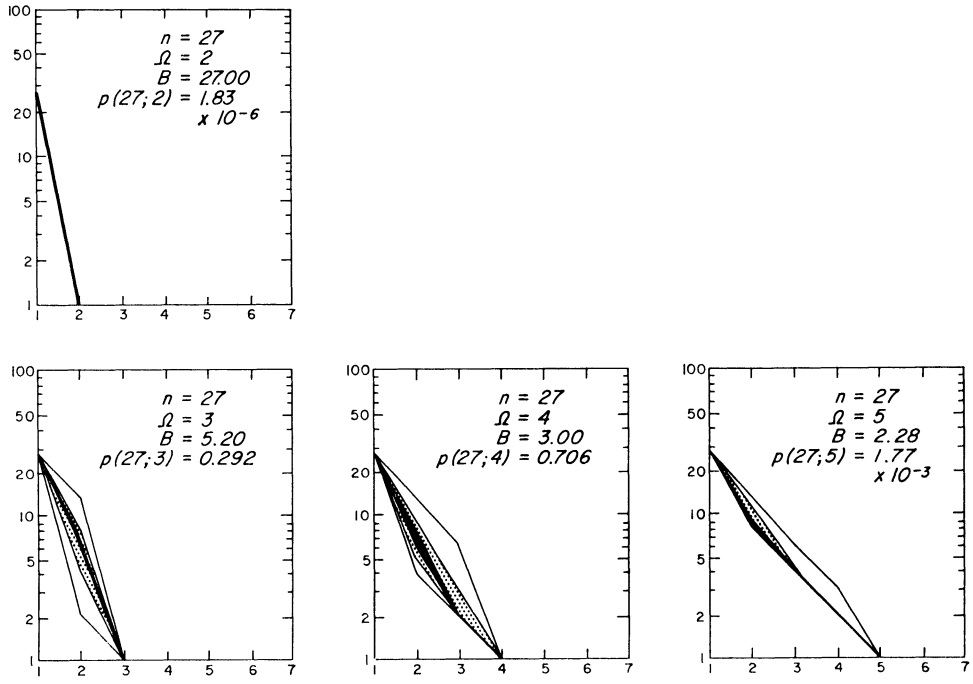

FIG. 9.—Statistics of population of topologically random channel networks with 27 first-order streams. Vertical coordinate is Strahler stream number (logarithmic scale); horizontal coordinate is stream order. Black area contains most probable 50 per cent of networks. Black and dotted areas together contain most probable 95 per cent of networks. Outer parallelogram contains all possible networks.

The fact that for every stream of given order, except the first, there must be at least two streams of the next lower order means that on a Horton diagram the points for any channel network with given order and given number of first-order streams will necessarily lie within a relatively restricted parallelogram-shaped region whose long diagonal is the locus of points which exactly satisfy

ber of topologically distinct networks $N[n_1, n_2, \ldots, n_{\Omega-1}, 1]$ given by (14) corresponding to each possible set of stream numbers by the total number $N(n;\Omega)$ given by (12) or by summation in order to obtain for specified n and Ω the normalized, or probability, distribution $p[n_1, n_2, \ldots, n_{\Omega-1}, 1]$ of the population of topologically random networks among the sets of stream numbers.

A flow diagram of this procedure, with main emphasis on the algorithm for systematically and exhaustively generating the sets of stream numbers, is given in figure 10. The distribution $p(n; \Omega)$ of the population of networks with specified n among the possible orders is found by dividing the number $N(n; \Omega)$ by the total number $N(n)$ given by (13) or by (11).

The computation was programmed in FORTRAN and executed on an IBM 7094 digital computer. Storage requirements were determined by means of (16). Do-it-yourself floating-point routines were required, because the large numbers involved (up to 10^{56}) exceeded the built-in capacity of the machine (only up to 10^{38}). The program is limited by size of memory to computations with n less than about 100.

Values of $N(n)$, $p(n; \Omega)$, and $D(n; \Omega)$ for various values of n and Ω and of $p[n_1, n_2, \ldots, n_{\Omega-1}, 1]$ for the various values of Ω when $n = 27$ are given in tables 2, 3, and 4.

The computations, whose results are displayed graphically in figures 9, 11, and 12, show that in a topologically random population of channel networks the most probable networks approximately obey Horton's geometric-series law, as speculated, but exhibit certain systematic deviations. For networks with given number n of first-order streams the most probable network order Ω is that which makes the geometric mean bifurcation ratio $B = n^{1/(\Omega-1)}$ closest to 4. For networks with both n and Ω given the most probable networks have the property that the bifurcation ratio of the second-order streams B_2 is always close to 4 and, hence, that the bifurcation ratios respectively decrease, remain unchanged, or increase with order and the corresponding curves on the Horton diagram are respectively concave upward, straight, or concave downward according as the geometric mean bifurcation ratio is less than, equal to, or greater than 4.

Comparison of these results with several typical natural networks is highly encouraging, as illustrated in figures 13 and 14.

STATISTICAL COMPARISON

The null hypothesis to be tested is that a random sample of sets of stream numbers from natural networks not subject to systematic inhomogeneity or anisotropy in bedrock or environment could have been drawn from a topologically random population. An appropriate statistical test for this hypothesis is the goodness-of-fit test described by Mood (1950, p. 270–271), inasmuch as the foregoing theoretical analysis provides the necessary expected frequencies.

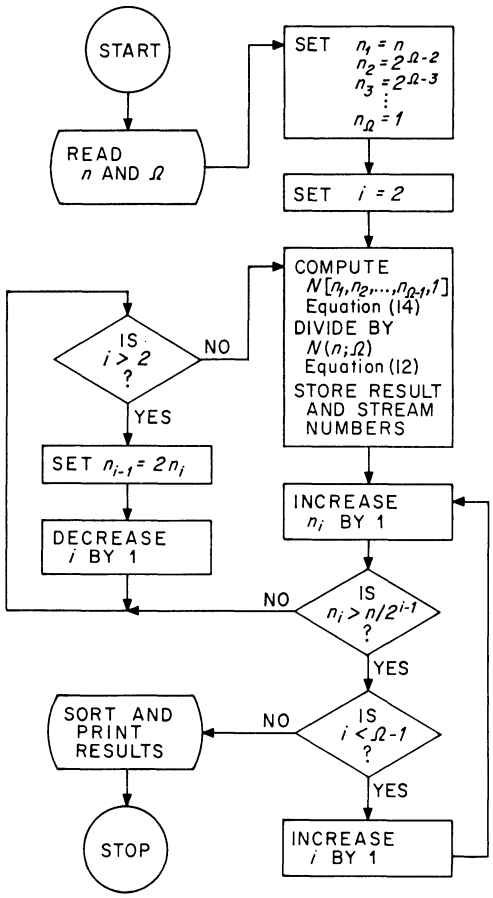

FIG. 10.—Flow diagram of procedure for computing law of stream numbers for population of topologically random channel networks with given number of first-order streams n and given order Ω. Main emphasis is on algorithm for systematically and exhaustively generating the sets of stream numbers.

The data available for analysis consist of 172 published sets of stream numbers selected from the 246 sets previously analyzed (references in table 1). All 152 third-order sets were admitted on the assumption that by virtue of their generally small size their basins would be relatively free of geologic controls. Only 18 fourth-order and 2 fifth-order sets were retained, the rest being rejected because of geologic controls, inadequate descriptions, or too large n. Because these 172 sets of stream numbers are in many cases from basins selected on the basis of size or, more frequently, order, they do not collectively constitute a random sample, but must be subdivided into separate samples corresponding to the various combinations of n and Ω. Whether these separate samples actually are random, as necessarily assumed in the following analysis, cannot be determined with certainty, because most authors vouchsafe few, if any, details about their method of selecting basins for investigation.

Unfortunately, nearly all of these samples are too small for meaningful statistical analysis, the majority consisting of just one set of stream numbers and only four, all third-order, containing 10 or more sets of stream numbers. Analysis of these four samples by means of the goodness-of-fit test indicates that, as hoped, they could have been drawn

TABLE 2

TOPOLOGICALLY DISTINCT NETWORKS

n	$N(n)$	$p(n;\Omega)$					
		$\Omega=2$	$\Omega=3$	$\Omega=4$	$\Omega=5$	$\Omega=6$	$\Omega=7$
6	42	0.381	0.619	.....	.....	.....	.....
16	9.695×10^{6}	$.169\times10^{-2}$	.753	0.245	0.103×10^{-6}	.....	.....
25	1.290×10^{12}	$.650\times10^{-5}$	.357	.642	$.772\times10^{-3}$	.....	.....
27	1.837×10^{13}	$.183\times10^{-5}$	.292	.706	$.177\times10^{-2}$	.....	.....
50	5.096×10^{26}	$.552\times10^{-12}$	$.194\times10^{-1}$	.874	.106	0.243×10^{-7}	.....[4]
64	9.430×10^{34}	$.489\times10^{-16}$	$.307\times10^{-2}$	.743	.254	$.382\times10^{-5}$	0.106×10^{-3}
75	3.115×10^{41}	$.303\times10^{-19}$	$.683\times10^{-3}$	.617	.382	$.437\times10^{-4}$	$.347\times10^{-2}$
100	2.275×10^{56}	0.139×10^{-26}	0.201×10^{-4}	0.361	0.638	0.122×10^{-2}	0.115×10^{-1}
200	1.29×10^{116}	.....	.....	.....	.....	.....	.....
500	1.35×10^{296}	.....	.....	.....	.....	.....	.....
1000	5.12×10^{596}	.....	.....	.....	.....	.....	.....

TABLE 3

SETS OF STREAM NUMBERS

n	$D(n;\Omega)$							
	$\Omega=3$	$\Omega=4$	$\Omega=5$	$\Omega=6$	$\Omega=7$	$\Omega=8$	$\Omega=9$	$\Omega=10$
6	2	.....	.....	.....	.....	.....	.....	.....
16	7	9	1	.....	.....	.....	.....	.....
25	11	25	10	.....	.....	.....	.....	.....
27	12	30	14	.....	.....	.....	.....	.....
50	24	132	190	46	.....	.....	.....	.....
64	31	225	455	201	1	.....	.....	.....
75	36	306	744	436	14	.....	.....	.....
100	49	576	2024	1980	284	.....	.....	.....
200	99	2401	18424	46420	32694	2790	.....	.....
500	249	15376	3.101×10^{5}	2.231×10^{6}	5.680×10^{6}	4.388×10^{6}	5.436×10^{6}	.....
1000	499	62001	2.542×10^{6}	3.814×10^{7}	2.159×10^{8}	4.419×10^{8}	2.693×10^{8}	2.287×10^{7}

TABLE 4

TOPOLOGICALLY DISTINCT NETWORKS WITH 27 FIRST-ORDER STREAMS

n_1	n_2	n_3	n_4	n_5	$p[\ldots]$	Sum	n_1	n_2	n_3	n_4	n_5	$p[\ldots]$	Sum
27	1	....	....	....	1.000	1.000	27	11	3	1	....	0.106×10^{-2}	0.999
							27	11	2	1	....	$.604\times10^{-3}$	1.000
27	6	1	....	....	0.319	0.319	27	11	4	1	....	$.353\times10^{-3}$	.....
27	7	1	....	....	.254	.573	27	12	3	1	....	$.382\times10^{-4}$	.....
27	5	1	....	....	.211	.784	27	10	5	1	....	$.380\times10^{-4}$	.....
27	8	1	....	....	.109	.892	27	12	4	1	....	$.191\times10^{-4}$	.....
27	4	1	....	....	$.691\times10^{-1}$	.962	27	11	5	1	....	$.189\times10^{-4}$	.....
27	9	1	....	....	$.249\times10^{-1}$	.987	27	12	2	1	....	$.164\times10^{-4}$	.....
27	3	1	....	....	$.988\times10^{-2}$	.996	27	12	5	1	....	$.204\times10^{-5}$	.....
27	10	1	....	....	$.293\times10^{-2}$	0.999	27	13	3	1	....	$.326\times10^{-6}$	.....
27	2	1	....	....	$.468\times10^{-3}$	1.000	27	13	4	1	....	$.228\times10^{-6}$	.....
27	11	1	....	....	$.162\times10^{-3}$	.....	27	13	2	1	....	$.109\times10^{-6}$	.....
27	12	1	....	....	$.351\times10^{-5}$	.....	27	13	5	1	....	$.407\times10^{-7}$	.....
27	13	1	....	....	0.191×10^{-7}	.....	27	12	6	1	....	$.227\times10^{-7}$	.....
							27	13	6	1	....	0.136×10^{-8}	.....
27	7	2	1	....	0.263	0.263							
27	6	2	1	....	.198	.461	27	9	4	2	1	0.451	0.451
27	8	2	1	....	.169	.630	27	8	4	2	1	.281	.733
27	8	3	1	....	$.845\times10^{-1}$	.715	27	10	4	2	1	.212	.945
27	7	3	1	....	$.657\times10^{-1}$	.780	27	11	4	2	1	$.352\times10^{-1}$	.980
27	5	2	1	....	$.656\times10^{-1}$	.846	27	10	5	2	1	$.114\times10^{-1}$	.992
27	9	2	1	....	$.542\times10^{-1}$	.900	27	11	5	2	1	$.566\times10^{-2}$	.997
27	9	3	1	....	$.452\times10^{-1}$	.946	27	12	4	2	1	$.191\times10^{-2}$	0.999
27	6	3	1	....	$.165\times10^{-1}$	.962	27	12	5	2	1	$.612\times10^{-3}$	1.000
27	10	3	1	....	$.106\times10^{-1}$	.973	27	13	4	2	1	$.228\times10^{-4}$	.....
27	10	2	1	....	$.851\times10^{-2}$	.981	27	12	6	2	1	$.136\times10^{-4}$	.....
27	4	2	1	....	$.716\times10^{-2}$	.988	27	13	5	2	1	$.122\times10^{-4}$	.....
27	9	4	1	....	$.452\times10^{-2}$	.993	27	12	6	3	1	$.113\times10^{-5}$	.....
27	8	4	1	....	$.282\times10^{-2}$	.996	27	13	6	2	1	$.814\times10^{-6}$	.....
27	10	4	1	....	0.213×10^{-2}	0.998	27	13	6	3	1	0.678×10^{-7}	.....

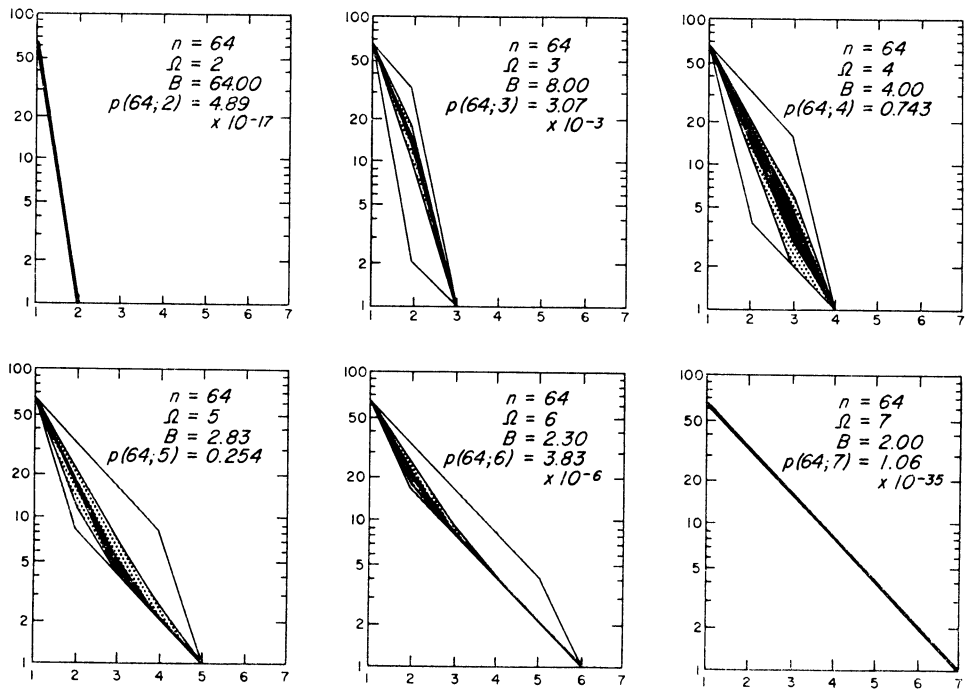

FIG. 11.—Statistics of population of topologically random channel networks with 64 first-order streams. Coordinates and symbols same as in fig. 9.

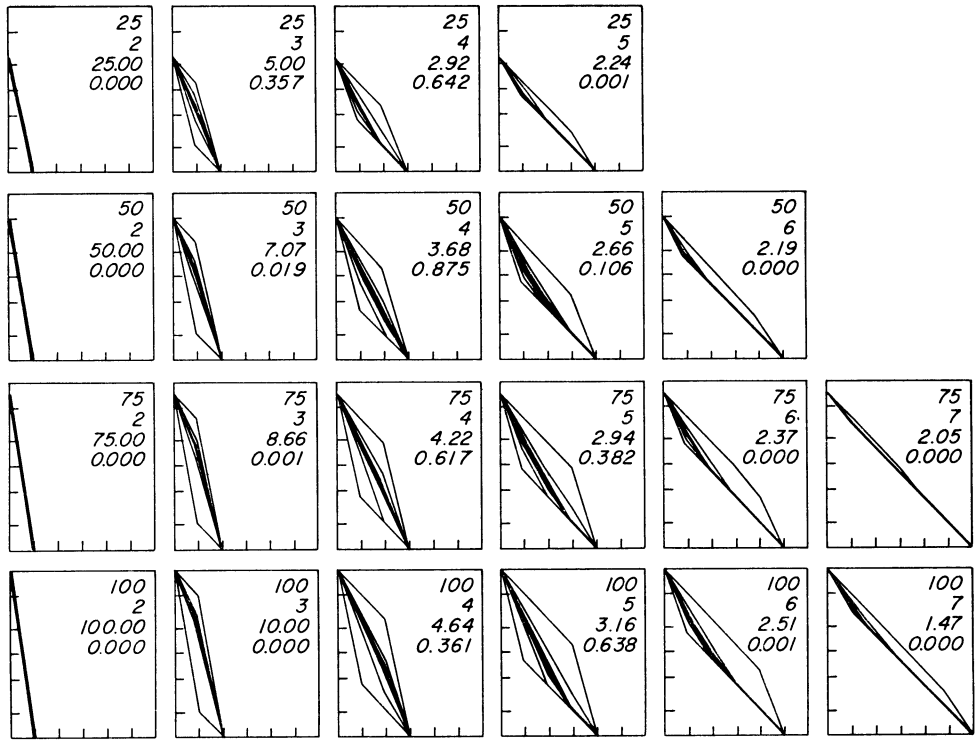

FIG. 12.—Statistics of four populations of topologically random channel networks with 25, 50, 75, and 100 first-order streams, respectively, for comparison. Coordinates and symbols same as in fig. 9, except dotted pattern omitted. The four numbers in each diagram are, from top to bottom, n, Ω, B, and $p(n;\Omega)$.

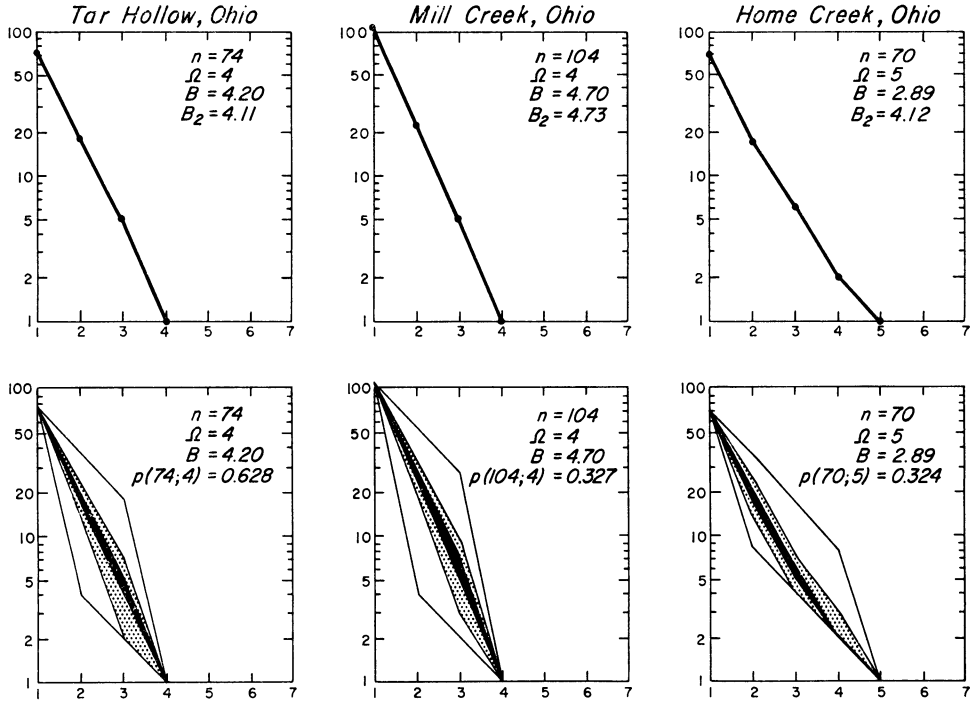

FIG. 13.—Three natural channel networks with similar and relatively homogeneous and isotropic bedrock and environment compared with corresponding topologically random populations (data from Morisawa, 1962, p. 1036).

from a topologically random population, for in none of the four cases was the null hypothesis rejected at the .05 significance level. Details and references are given in tables 5 and 1, respectively. In each case the χ^2 approximation was used (Mood, 1950, p. 259, 271, 424), and the effect of sampling without replacement was assumed negligible.

In a somewhat less straightforward approach designed to utilize all of the available data, the topologically random population was divided into four equal groups, from the most probable 25 per cent of the networks to the least probable, and the 172 observed sets of stream numbers classified accordingly. Networks with probabilities overlap-

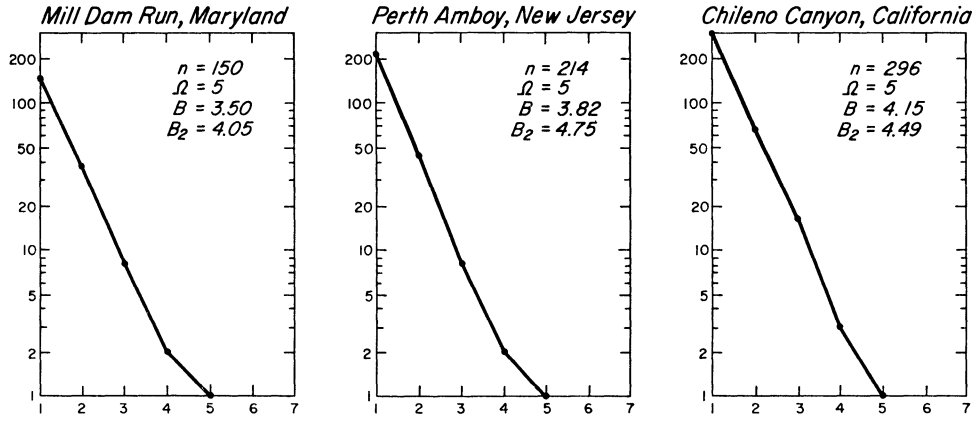

FIG. 14.—Three natural channel networks with dissimilar but relatively homogeneous and isotropic bedrock and environment (data from Schumm, 1956, p. 605). Just as in a topologically random population the concavity of the curves decreases practically to zero as B increases approximately to 4.

TABLE 5

STATISTICAL ANALYSIS

n_1	n_2	n_3	Observed	Expected	Likelihood Ratio λ	$-2 \ln \lambda^*$	Decision at .05
9	2	1	10	7.31	0.233	2.91	Accept
9	3	1	4	6.09		(2, 5.99)	
9	4	1	0	0.60			
11	2	1	6	3.56	.279	2.55	Accept
11	3	1	4	6.24		(3, 7.81)	
11	4	1	2	2.08			
11	5	1	0	0.12			
13	2	1	1	1.69	.367	2.01	Accept
13	3	1	7	5.08		(4, 9.49)	
13	4	1	2	3.56			
13	5	1	1	0.64			
13	6	1	0	0.02			
14	2	1	4	1.19	0.030	7.04	Accept
14	3	1	3	4.46		(5, 11.1)	
14	4	1	4	4.16			
14	5	1	0	1.11			
14	6	1	0	0.08			
14	7	1	0	0.00			

* Figures in parentheses are number of degrees of freedom for χ^2 distribution and critical value for rejection at .05 significance level, respectively.

ping two or more groups were divided proportionally, an observed network with the stream numbers [27, 8, 2, 1], for example, being counted as 0.23 in the second most probable group and 0.77 in the third (see table 4 for the relevant probabilities). The goodness-of-fit test was then applied to the observed frequencies in the four classes, which by their construction have equal expected frequencies. The null hypothesis, that the sample could have been drawn from a topologically random population, was not rejected at the .05 significance level for networks free of geologic controls; but it was rejected for fourth-order networks selected without regard to such controls. Details and references are given in table 6. As before, in each case the χ^2 approximation was used, and the effect of sampling without replacement was assumed negligible.

These statistical comparisons strongly support the conclusion that populations of natural channel networks developed in the absence of geologic controls are topologically random (which does not, however, imply that the lengths, shapes, and orientations of the links necessarily are also random) and therefore that, as proposed, the law of stream numbers is indeed largely a consequence of random development of the topology of channel networks according to the laws of chance.

TABLE 6

MODIFIED STATISTICAL ANALYSIS

Networks; Order	Geologic Controls	Observed	Expected	Likelihood Ratio λ	$-2 \ln \lambda^*$	Decision at .05
172; 3, 4, 5	No[1,2,3]	34.92 38.09 47.33 51.66	43 43 43 43	0.118	4.26 (3, 7.81)	Accept
152; 3	No[1]	32.44 34.33 41.37 43.86	38 38 38 38	.305	2.37 (3, 7.81)	Accept
18; 4	No[2]	2.48 3.76 3.96 7.80	4.5 4.5 4.5 4.5	.196	3.26 (3, 7.81)	Accept
60; 4	Yes[4]	8.44 6.31 16.38 28.87	15 15 15 15	0.000	20.03 (3, 7.81)	Reject

* Figures in parentheses are number of degrees of freedom for χ^2 distribution and critical value for rejection at .05 significance level, respectively.

[1] Data for third-order networks same as in table 1.

[2] Data for fourth-order networks from Schumm (1956, p. 606), Melton (1957, table 2, facing p. 88), Smith (1958, p. 1003), Maxwell (1960, p. 79–82), and Morisawa (1962, p. 1026–1027, 1036–1037).

[3] Data for fifth-order networks from Maxwell (1960, p. 81) and Morisawa (1962, p. 1026, 1036).

[4] Data same as in table 1, except networks with n greater than about 100 omitted.

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